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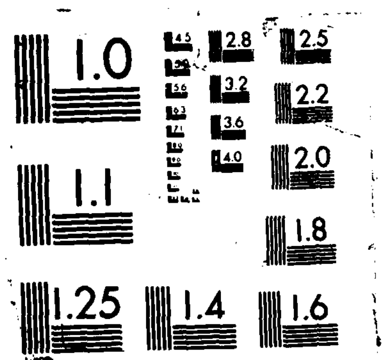
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Identifiability of Multivariate ARMA Models

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This paper proved that multivariate ARMA models ~~is~~ are identifiable. Some properties of Multivariate ARMA models were given.

(Keywords: matrix coefficient polynomials; stationary; holomorphic; white noise.)

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This paper proved that a multivariate ARMA model is identifiable. Some properties of Multivariate ARMA models were given.

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1 Introduction

First we impose the condition:

Two matrix coefficient polynomials

$$A(z) = \sum_{k=0}^{p} A_k z^k \quad \text{and} \quad B(z) = \sum_{j=0}^{q} B_j z^j$$

(where A_k, B_j are $r \times r$ matrices) are said to have no common left divisors, if matrix coefficient polynomial $D(z) = \sum_{i=0}^{t} D_i Z^i$ is such that

$$A(z) = D(z)A_1(z) \quad \text{and} \quad B(z) = D(z)B_1(z)$$

then $\det(D(z))$ is a constant.

A multivariate stationary process $X_t = (X_1(t), X_2(t), \cdots, X_r(t))^\tau$ is said to follow a multivariate ARMA model if it can be expressed in the form

$$\sum_{k=0}^{p} A_k X_{t-k} = \sum_{j=0}^{q} B_j \varepsilon_{t-j}, \quad t = 0, \pm 1, \pm 2, \cdots \qquad (1.1)$$

where $\varepsilon_t = (\varepsilon_1(t), \varepsilon_2(t), \cdots, \varepsilon_r(t))^\tau$, $t = 0, \pm 1, \pm 2, \cdots$ is a multivariate white noise process. $E\varepsilon_t = 0$, $E\varepsilon_t \varepsilon_s^\tau = \delta_{t.s} I$. $A_0, A_1, \cdots, A_p, B_0, B_1, \cdots, B_q$ are $r \times r$ real matrices, $A_0 = I$. B_0 is positive definite and

a. $A(z) = \sum_{k=0}^{p} A_k z^k$ and $B(z) = \sum_{j=0}^{q} B_j z^j$ have no common left divisors.

b. $\det(A(z)) \neq 0$, $|z| \leq 1$; $\det(B(z)) \neq 0$, $|z| < 1$.

In this case (X_t) is said to be a multivariate ARMA series A multivariate ARMA series is said to be a multivariate AR(MA)

series if $q = 0$ ($p=0$).

Let B be the backward shift operator, we can express ARMA model (1.1) in the form

$$A(B)X_t = B(B)\varepsilon_t, \quad t = 0, \pm 1, \pm 2, \cdots \tag{1.2}$$

One of the basic problem associated with the multivariate ARMA models is the identification of the structure of (1.1), given the covariance function of $\{X_t\}$, by identification we mean here the following problem: given that $\{X_t\}$ conforms to some multivariate ARMA model of unspecified orders can we determine the values of p and q and the matrices $A_1, A_2, \cdots, A_k, B_0, B_1, \cdots, B_q$ uniquely from the covariances of $\{X_t\}$ [1]. In the case of $r = 1$, it is known that we can determine the values p, q and $A_1, A_2, \cdots, A_p$, $B_0, B_1, \cdots, B_q$ uniquely from the covariance function of $\{X_t\}$. So univariate ARMA model is identifiable.

It is easy to see that multivariate ARMA model is not identifiable. Suppose that, in model (1.1), $\det(A(z))$ and $\det(B(z))$ are all nonzero constant, then the following three models

$$A(B)X_t = B(B)\varepsilon_t$$

$$X_t = A^{-1}(B)B(B)\varepsilon_t$$

$$B^{-1}(B)A(B)X_t = \varepsilon_t \qquad t=0, \pm 1, \pm 2, \cdots$$

have the same stationary solution.

In fact, corresponding to a given covariance structure of a multivariate ARMA series there will be an "equivalent class" of models, and the problem of identifiability then becomes one of devising a

set of rules which select a unique representative model from each equivalent class. Without the solution of this problem, it will be difficult to consider the estimation problem of multivariate ARMA models. This problem has been studied principally by Hannan [2,3], but so far no result is satisfiable, because the set of rules given is not easy to verify. A reasonable and simple rule on the unique representative model is given by this paper.

2. Basic Theorems

A multivariate stationary process (X_t) is said to follow a generalized multivariate ARMA model if it can be expressed in the form

$$\sum_{k=0}^{p} A_k X_{t-k} = \sum_{j=0}^{q} B_j \varepsilon_{t-j}, \quad t = 0, \pm 1, \pm 2, \cdots . \tag{2.1}$$

where $\varepsilon_t = (\varepsilon_1(t), \varepsilon_2(t), \cdots, \varepsilon_r(t))^\tau$ is multivariate white noise process: $E\varepsilon_t = 0$, $E\varepsilon_t \varepsilon_s^\tau = \delta_{t.s} I$, $A_0, A_1, A_2, \cdots, A_p$, $B_0, B_1, \cdots, B_q$ are $r \times r$ real matrices. $A_0 = I$, B_0 is positive definite.

Let

$$A(z) = \sum_{k=0}^{p} A_k z^k , \qquad B(z) = \sum_{j=0}^{q} B_j z^j .$$

Theorem 2.1

If $\det(A(z)) \neq 0$, $|z|=1$, then model (2.1) has unique stationary solution.

Proof: Let us use the same signs given by Rozanov [4]. Write

$$B_k = \begin{bmatrix} b_1(k) \\ b_2(k) \\ \vdots \\ b_r(k) \end{bmatrix}, \quad k = 1,2,\cdots,q, \tag{2.2}$$

define

$$\varepsilon_t(b_i(k)) = b_i(k)\varepsilon_t, \quad i = 1,2,\cdots,r \tag{2.3}$$

$$S_k(\lambda) = \begin{bmatrix} E_\lambda \varepsilon_0(b_1(k)) \\ E_\lambda \varepsilon_0(b_2(k)) \\ \vdots \\ E_\lambda \varepsilon_0(b_r(k)) \end{bmatrix}, \quad k = 0,1,2,\cdots,q. \tag{2.4}$$

where E_λ is the spectral operator.

Since all the elements of $A^{-1}(e^{-i\lambda})$ are continuous in $\lambda \in [-\pi,\pi]$.

We can define

$$X_t = \sum_{k=0}^{q} \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})e^{i(t-k)\lambda}\, ds_k(\lambda). \tag{2.5}$$

$$t = 0, \pm 1, \cdots.$$

Where $i = \sqrt{-1}$.

$$EX_t = 0$$

$$E(X_t X_s^\tau) = (X_t, X_s)$$

$$= \sum_{k=0}^{q} \sum_{j=0}^{q} \left[\int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})e^{i(t-k)\lambda} dS_k(\lambda), \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})e^{i(s-j)\lambda} dS_j(\lambda) \right]$$

$$= \sum_{k=0}^{q} \sum_{j=0}^{q} \left\{ \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})e^{i(t-s)\lambda} dS_k(\lambda), \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})e^{i(k-j)\lambda} dS_j(\lambda) \right\}$$

$$= R_x(t-s).$$

It follows that $\{X_t\}$ is a r-variate stationary process with zero mean, and

$$\sum_{k=0}^{p} A_k X_{t-k} = \sum_{k=0}^{p} \sum_{j=0}^{q} \int_{-\pi}^{\pi} A_k A^{-1}(e^{-i\lambda}) e^{i(t-k-j)\lambda} dS_j(\lambda)$$

$$= \sum_{j=0}^{q} \int_{-\pi}^{\pi} \sum_{k=0}^{p} A_k e^{-ik\lambda} \; A^{-1}(e^{-i\lambda}) e^{i(t-j)\lambda} dS_j(\lambda)$$

$$= \sum_{j=0}^{q} \int_{-\pi}^{\pi} e^{i(t-j)\lambda} dS_j(\lambda)$$

$$= \sum_{j=0}^{q} B_j \varepsilon_{t-j}.$$

If $\{Y_t\}$ is also a stationary solution of model (2.1), write

$$\xi(\lambda) = \begin{bmatrix} E_\lambda Y_1(0) \\ E_\lambda Y_2(0) \\ \vdots \\ E_\lambda Y_r(0) \end{bmatrix} \tag{2.6}$$

where $(Y_1(t), Y_2(t), \cdots, Y_r(t))^\tau = Y_t$.

For any t

$$\int_{-\pi}^{\pi} \sum_{j=0}^{p} A_j e^{i(t-j)\lambda} d\xi(\lambda)$$

$$= \sum_{j=0}^{p} A_j Y_{t-j}$$

$$= \sum_{j=0}^{q} B_j \varepsilon_{t-j}$$

$$= \int_{-\pi}^{\pi} \sum_{j=0}^{q} e^{i(t-j)\lambda} dS_j(\lambda).$$

Let $A^{-1}(e^{-i\lambda}) = \sum_{m=-\infty}^{\infty} V_m e^{-im\lambda}$ be series expansion of $A^{-1}(e^{-i\lambda})$, then every elements of V_m tends to zero by negative exponential ratio as $|m| \longrightarrow \infty$ [5]. So, it follows that

$$Y_t = \int_{-\pi}^{\pi} e^{it\lambda} d\xi(\lambda) = \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda}) \sum_{j=0}^{q} e^{i(t-j)\lambda} dS_j(\lambda)$$

$$= X_t. \qquad t = 0, \pm 1, \pm 2, \cdots.$$

Corollary 2.1. If $\det(A(z)) \neq 0$, $|z|=1$, the unique stationary solution of (2.1) is in the form

$$X_t = \sum_{m=-\infty}^{\infty} \Lambda_m \varepsilon_{t-m} . \tag{2.8}$$

where Λ_m are $r \times r$ real matrices determined by

$$A^{-1}(z)B(z) = \sum_{m=-\infty}^{\infty} \Lambda_m z^m. \quad r_1 \le |z| \le r_2 \tag{2.9}$$

with $r_1 < 1$, $r_2 > 1$.

Proof: Suppose that $A^{-1}(z) = \sum_{m=-\infty}^{\infty} \Lambda_m z^m$, $r_1 \le |z| \le r_2$ according to (2.9).

$$\Lambda_m = \sum_{0}^{q} V_{m-j} B_j \tag{2.10}$$

since

$$A(z) \sum_{m=-\infty}^{\infty} V_m z^m = I \tag{2.11}$$

so

$$\sum_{k=0}^{p} A_k V_{m-k} = \delta_{0,m} I. \tag{2.12}$$

Every elements of V_m tends to zero by negative exponential ratio, so does that of Λ_m, as $|m| \longrightarrow \infty$, and it follows that $\{X_t\}$ given by (2.8) is a stationary process with zero mean, and

$$\sum_{k=0}^{p} A_k X_{t-k} = \sum_{k=0}^{p} A_k \sum_{m=-\infty}^{\infty} \Lambda_m \varepsilon_{t-k-m}$$

$$= \sum_{k=0}^{p} A_k \sum_{m=-\infty}^{\infty} \sum_{j=0}^{q} V_{m-j} B_j \varepsilon_{t-k-m}$$

$$= \sum_{k=0}^{p} A_k \sum_{m=-\infty}^{\infty} \sum_{j=0}^{q} V_{m-k-j} B_j \varepsilon_{t-m}$$

$$= \sum_{j=0}^{q} \sum_{m=-\infty}^{\infty} \left[\sum_{k=0}^{p} A_k V_{m-k-j} \right] B_j \varepsilon_{t-m}$$

$$= \sum_{j=0}^{q} B_j \varepsilon_{t-j} .$$

Corollary 2.2. If $\det(A(z)) \neq 0$, when $|z|=1$, then

a. the unique stationary solution of (2.1) is in the form

$$X_t = \sum_{m=0}^{\infty} \Lambda_m \varepsilon_{t-m} \tag{2.13}$$

if and only if every elements of $A^{-1}(z)B(z)$ is holomorphic function of z in the field of $(z, |z|<1)$. If $\det(A(z))$ and $\det(B(z))$ have no common divisor,

the condition is equivalent to that all the roots of $\det(A(z))$ are outside the unit circle.

b. the unique stationary solution of (2.1) is in the form

$$X_t = \sum_{M=-\infty}^{M} \Lambda_m \varepsilon_{t-m} \tag{2.14}$$

if and only if every elements of $z^{-m}A^{-1}(z)B(z)$ is holomorphic function of z in the field of $\{z, |z| \geq 1$ including the infinite point$\}$. If $\det(A(z))$ and $\det(B(z))$ have no common divisors, the condition is equivalent to that all roots of $\det(A(z))$ are inside the unit circle.

c. the unique stationary solution of (2.1) is in the form

$$X_t = \sum_{m=-\infty}^{\infty} \Lambda_m \varepsilon_{t-m} \tag{2.15}$$

if and only if all the elements of $A^{-1}(z)B(z)$ are holomorphic functions of z in the field of $\{z, r_1 \leq |z| \leq r_2\}$ where $r_1 < 1, r_2 > 1$. If $\det(A(z))$ and $\det(B(z))$ have no common divisors, the condition is equivalent to that all roots of $\det(A(z))$ scatter both outside and inside the unit circle.

(The proof is erased, because it is a problem of algebra)

Corollary 2.3 Under the same condition of Theorem 2.1, model (2.1) [redacted] and model

$$\sum_{k=0}^{s} \Phi_k Y_{t-k} = \sum_{j=0}^{\ell} \Psi_j \varepsilon_{t-j}, \quad t=0,\pm 1,\pm 2,\cdots \tag{2.16}$$

(where $\Phi_0,\Phi_1,\cdots,\Phi_s,\Psi_1,\Psi_2,\cdots,\Psi_t$ are $r \times r$ real matrices. $\Phi_0=I$, Ψ_0 is positive definite, $\det\left[\sum_{k=0}^{s} \Phi_k z^k\right] \neq 0$, $|z|=1$)

have same stationary solution if and only if

$$A^{-1}(z)B(z) = \Phi^{-1}(z)\Psi(z), \quad |z|\le 1 \tag{2.17}$$

where

$$\Phi(z) = \sum_{k=0}^{s} \Phi_k z^k, \quad \Psi(z) = \sum_{j=0}^{\ell} \Psi_j z^j$$

Proof is erased.

Theorem 2.2 If multivariate stationary process $\{X_t\}$ follows multivariate ARMA model (1.1), then the white noise process $\{\varepsilon_t\}$ is the innovation process of $\{X_t\}$ [6], and $B_0 B_0^\tau$ is the one step prediction error matrix, i.e.

$$B_0 = \left\{(X_t - \mathrm{Proj}_{H_x(t-1)} X_t, X_t - \mathrm{Proj}_{H_x(t-1)} X_t\right\}^{1/2} \tag{2.18}$$

$$\varepsilon_t = B_0^{-1}\left[X_t - \mathrm{Proj}_{H_x(t-1)} X_t\right] \tag{2.19}$$

where $H_x(t-1)$ is the Hilbert space extended by $\{X_{t-1}, X_{t-2},\cdots\}$.

Proof: $A(z)$ is holomorphic inside the unit circle, the stationary solution of (1.1) is in the form

$$X_t = \sum_{m=0}^{\infty} \Lambda_m \int_{-\pi}^{\pi} e^{i(t-m)\lambda} dE_\lambda \varepsilon_0$$

$$= \int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})B^{-1}(e^{-i\lambda})e^{it\lambda}dE_{\lambda}\varepsilon_0$$

$$(X_t, X_s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A^{-1}(e^{-i\lambda})B(e^{-i\lambda})(A^{-1}(e^{-i\lambda})B(e^{-i\lambda}))^* e^{i(t-s)\lambda}d\lambda$$

where A^* denotes the conjugate transpose matrix of A. It follows that the spectral density matrix function of $\{X_t\}$ is

$$f(\lambda) = \frac{1}{2\pi} A^{-1}(e^{-i\lambda})B(e^{-i\lambda})(A^{-1}(e^{-i\lambda})B(e^{-i\lambda}))^* \tag{2.20}$$

Let $C(e^{-i\lambda}) = A^{-1}(e^{-i\lambda})B(e^{-i\lambda}) = \sum_{j=0}^{\infty} \Lambda_m e^{-im}.$ (2.21)

$c_0, c_1, \cdots,$ be the Wold coefficient matrices of $\{X_t\}$ and

$$\Gamma(z) = \sum_{j=0}^{\infty} C_j z^j, \quad |z| \le 1 \tag{2.22}$$

then $f(\lambda) = \frac{1}{2\pi} \Gamma(e^{-i\lambda})\Gamma^*(e^{-i\lambda})$. [6] (2.23)

Now what we need to do is to prove $C(z) = \Gamma(z)$. According to (2.21), (2.23) and [6], we only need to prove

$$\Gamma(0)\Gamma^*(0) = C(0)C^*(0) \tag{2.24}$$

Since $\Gamma(0)\Gamma^*(0) \ge C(0)C^*(0)$[6] (2.25)

$\det(C(z))$, $\det(\Gamma(z))$ are all the maximum function in the $H_{r/2}$ space [7.8] with the same boundary value on the unit circle, so

$$\det(\Gamma(0)\Gamma^*(0) = \det(C(0)C^*(0) \tag{2.26}$$

If V is an invertible matrix such that

$$V\Gamma(0)\Gamma^*(0)V^* = I \ge VC(0)C^*(0)V^* \tag{2.27}$$

and U is a unitary matrix such that

$$I = UU^* \geq UVC(0)C^*(0)V^*U^* = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r \end{bmatrix} \qquad (2.28)$$

we know $0 \leq \lambda_i \leq 1$, according to (2.26), we have $\lambda_1\lambda_2\cdots\lambda_r = 1$,

and $\lambda_1=\lambda_2=\cdots=\lambda_r=1$, therefore

$$\Gamma(0)\Gamma^*(0) = C(0)C^*(0).$$

It follows that

$$C(z) = \Gamma(z), \ |z| \leq 1.$$

Let $\{\xi_n\}$ be the innovation process of $\{X_t\}$ with $E\xi_n\xi_n^\tau = I$.

$$X_t = \sum_{j=0}^{\infty} C_j \xi_{t-j} = \sum_{j=0}^{\infty} \Lambda_j \xi_{t-j} \qquad t=0,\pm1,\pm2,\cdots$$

be the Wold decomposition of $\{X_t\}$, using $\varepsilon_t \perp H_x(t-1)$,

$\xi_t=\Lambda_0^{-1}(X_t-\mathrm{Proj}_{H_x(t-1)}X_t)$, it follows that

$$(\varepsilon_t,\xi_t) = \Lambda_0\Lambda_0^{-1} = I,$$

and therefore

$$\varepsilon_t = \xi_t \quad , \quad t=0,\pm1,\pm2,\cdots.$$

Because $A_0 = 1$, so $B_0 = \Lambda_0$, $B_0B_0^2 = \Lambda_0\Lambda_0^\tau$.

Corollary 2.4. Let stationary series $X_t=(X_1(t),X_2(t),\cdots,X_r(t))^\tau$ follows multivariate ARMA model (1.1), stationary series $Y_t = (Y(t),Y_2(t)\cdots,Y_r(t))$ follows multivariate ARMA model

$$\sum_{k=0}^{s} \Phi_k Y_{t-k} = \sum_{j=0}^{\ell} \Psi_j \varepsilon_{t-j} \qquad (2.29)$$

then

$$E(X_tX_s^\tau) = E(Y_tY_s^\tau), \quad \text{for all } t,s=0,\pm1,\pm2,\cdots$$

if and only if

$$A^{-1}(z)B(z) = \Phi^{-1}(z)\Psi(z) \qquad |z|\le 1 \qquad (2.30)$$

where $$\phi(z) = \sum_{j=0}^{s}\Phi_k z^k, \quad \Psi(z) = \sum_{k=0}^{\ell}\Psi_j z^j .$$

Proof: If $A^{-1}(z)B(z) = \Phi^{-1}(z)\Psi(z)$, then $\{X_t\}$, $\{Y_t\}$ have the same spectral density matrix function (2.20), and therefore, $\{X_t\},\{Y_t\}$ have the same covariance structure.

If $\{X_t\}$, $\{Y_t\}$ have same covariance structure, they have same Wold coefficients matrices $C_0, C_1, C_2, \cdots,$ and so

$$A^{-1}(z)B(z) = \Phi^{-1}(z)\Psi(z) = \Gamma(z) = \sum_{j=0}^{\infty} C_j z^j, \ |z|\le 1.$$

3. Identifiability of Multivariate ARMA models

Theorem 3.1 Assume stationary series $X_t = (X_1(t), X_2(t), \cdots, X_r(t))^\tau$ follows multivariate ARMA model (1.2) and where $\det(A(z))$ and $\det(B(z))$ have no common divisors, then if

$$\det(A_p) \ne 0 \qquad (\text{or } \det(B_q)\ne 0) \qquad (3.1)$$

We can determine the values of p, q and the matrices $A_0, A_1, \cdots, A_p, B_0, B_1, \cdots, B_q$ uniquely from the covariance structure of $\{X_t\}$.

Proof: Assume that X_t follows another multivariate ARMA model

$$\sum_{j=0}^{s}\phi_k X_{t-k} = \sum_{j=0}^{\ell}\Psi_j \varepsilon_{t-j} \qquad (3.2)$$

with $\det(\Phi(z))$ and $\det(\Psi(z))$ have no common divisors and

$\det(\Phi_s) \neq 0$ (or $\det(\Psi_\ell) \neq 0$).

Using Corollary 2.4, we have

$$A^{-1}(z)B(z) = \Phi^{-1}(z)\Psi(z) \ , \ |z| \le 1. \tag{3.3}$$

Let $\tilde{A}(z)$ and $\tilde{B}(z)$ be the adjoint matrices of $A(z)$ and $B(z)$ respectively then we have

$$\frac{b(z)}{a(z)} \Phi(z)\tilde{A}(z) = \Psi(z)\tilde{B}(z) \qquad |z| \le 1$$

where $b(z) = \det(B(z))$, $a(z) = \det(A(z))$. Since $a(z), b(z)$ have no common divisor, it follows that

$$\frac{1}{a(z)} \Phi(z)\tilde{A}(z) = D(z)$$

must be a matrix coefficient polynomial and

$$\Phi(z) = D(z)A(z) \qquad |z| \le 1 \tag{3.4}$$

$$\Psi(z) = D(z)B(z) \qquad |z| \le 1 \tag{3.5}$$

Note, $\Phi(z)$, $\Psi(z)$ have no common left divisor, so

$$\det(D(z)) = \text{constant}.$$

Write $D(z) = \sum_{k=1}^{m} D_k z^k$, then $\det(D_m) = 0$, if $m \ge 1$.

Using (3.4), $\det(\phi_s) \neq 0$, $\det(A_p) \neq 0$, (or using (3.5), $\det(\Psi_1)$, $\det(B_q) \neq 0$), we have $m=0$, so

$$D(z) = D_0 \ \text{ is a constant matrix.}$$

From $\Phi_0 = A_0 = I$, we have $D_0 = I$, therefore

$$\Phi(z) = A(z), \quad \Psi(z) = B(z).$$

A familiar result about multivariate AR.MA model follows directly from Theorem 3.1. Multivariate AR.MA model are all identifiable.

Definition:

A stationary series $X_t = (X_1(t), X_2(t), \cdots, X_r(t))^\tau$ is said to follow a multivariate ARMA(LC) model, if it can be expressed in the form

$$\sum_{k=0}^{k} a_k X_{t-k} = \sum_{j=0}^{q} B_j \varepsilon_{t-j} \tag{3.6}$$

where

a) $\varepsilon_t = (\varepsilon_1(t), \varepsilon_2(t), \cdots, \varepsilon_r(t))^\tau$ is a multivariate white noise process. $E\varepsilon_t = 0$, $E\varepsilon_t \varepsilon_s^\tau = \delta_{s.t} I$.

b) $a_1, a_2, \cdots, a_p$ are real constant, $a_0 = 1$, $\sum_{j=0}^{p} a_j z^j \neq 0$, for $|z| \le 1$. $B_0, B_1, \cdots, B_q$ are $r \times r$ real matrices. B_0 is positive definite, $\det(B(z)) \neq 0$. $|z| < 1$.

$$B(z) = \sum_{j=1}^{q} B_j z^j = (b_{ik}(z))_{r \times r}$$

c) The set of polynomials $\{a(z), b_{ik}(z), i.k = 1.2 \cdots r\}$ have no common divisors.

It can be seen that the values of multivariate ARMA(LC) models are not more complex than that of multivariate ARMA models. But we can prove the following result.

Theorem 3.2 1. Any multivariate stationary series that follows some multivariate ARMA model will follow some multivariate ARMA(LC) model.

2. Multivariate ARMA(LC) model is identifiable.

Proof 1. Assume that X_t follows multivariate ARMA model (1.2). Let $a(z) = \det(A(z))$, $B_1(z) = \tilde{A}(z)B(z) = (d_{ij})_{r \times r}$ where

$\tilde{A}(z)$ is the adjoint matrix of $A(z)$. Using Corollary 2.3 we have that X_t can be expressed in the form

$$a(B)X_t = B_1(B)\varepsilon_t \tag{3.7}$$

where B is the backward shift operator.

Assume $f(z)$ is the maximum divisor of

$$\{a(z),\ d_{ij}(z);\quad i,j=1,2,\cdots,r\}$$

then

$$a(z) = a_1(z)f(z)$$

$$d_{ij}(z) = C_{ij}(z)f(z) \qquad i,j=1,\cdots,r$$

and

$\{a_1(z),\ c_{ij}(z);\quad i,j=1,2,\cdots,r\}$ have no common divisors.

Let

$$C(z) = (C_{ij}(z))_{r\times r}$$

it follows that X_t follows ARMA(LC) model

$$a_1(B)X_t = C(B)\varepsilon_t$$

Assume X_t is a multivariate stationary series that follows ARMA(LC) model

$$a(B)X_t = B(B)\varepsilon_t$$

and $\qquad \phi(B)X_t = \Psi(B)\zeta_t\ .$

According to Corollary 2.4

$$a^{-1}(z)B(z) = \phi^{-1}(z)\Psi(z), \quad |z|\le 1 \tag{3.8}$$

so $\qquad \phi(z)B(z) = a(z)\Psi(z), \quad |z|\le 1 \qquad (3.8)$

Assume $g(z)$ is the maximum divisor of the polynomials $a(z)$ and $\phi(z)$, and

$$\phi(z) = g(z)\phi_1(z)$$

$$a(z) = g(z)a_1(z), \quad g(0) = 1 \tag{3.9}$$

then, $$\phi_1(z)B(z) = a_1(z)\Psi(z)$$

and $\phi_1(z)$ is a common divisor of the set of polynomials of $\Psi(z)$. Therefore

$$B(z) = a_1(z)\left[\frac{1}{\phi_1(z)}\Psi(z)\right] = a_1(z)\Psi_1(z)$$

From (3.9), we see

$$a_1(z) = 1, \text{ and } \phi_1(z) = 1 \text{ all the same,}$$

therefore

$$a(z) = \phi(z)$$

$$B(z) = \Psi(z)$$

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